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ON UNILATERAL SHIFT OPERATORS AND C_0-OPERATORS

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Abstract

Let $S^{(n)}$ be a unilateral shift operator on a Hilbert space of multiplicity n. In this paper, we prove a generalization of the theorem that if $S^{(1)}$ is unitarily equivalent to an operator matrix form $\begin{pmatrix} S^{(1)} & * \\ 0 & E \end{pmatrix}$ relative to a decomposition $\mathscr{M} \oplus \mathscr{N}$, then E is in a certain class C_0 which will be defined below.

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Suppose $\mathscr{K}$ and $\mathscr{H}$ are separable Hilbert spaces and $\mathscr{L}(\mathscr{H}, \mathscr{K})$ is the algebra of all bounded linear operators from $\mathscr{H}$ into $\mathscr{K}$. In particular, let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. Throughout this paper we write $\mathbf{U}$ for the open unit disc in the complex plane $\mathbf{C}$ and $\mathbf{T}$ for the boundary of $\mathbf{U}$. The space $L^p = L^p(\mathbf{T})$, $1 \le p \le \infty$, is the usual Lebesgue function space. For $1 \le p \le \infty$, we denote by $H^p = H^p(\mathbf{T})$ the subspace of L^p consisting of those functions whose negative Fourier coefficients vanish. If $u \in H^\infty$, then we have a Fourier series

$$(1) \qquad u(e^{it}) = \sum_{n=0}^{\infty} a_n e^{int}.$$

Let T be a completely nonunitary contraction on a Hilbert space $\mathscr{H}$. Then

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for $u \in H^{\infty}$, we define a functional calculus

$$u(T) = \lim_{r \to 1} \sum_{n=0}^{\infty} a_n r^n T^n , \tag{2}$$

where the limit exists in the strong operator topology (cf. [1, p. 16]). A completely nonunitary contraction $T \in \mathcal{L}(\mathcal{H})$ is said to be of *class* C_0 if there exists a non-zero function $u \in H^{\infty}(\mathbf{T})$ such that the functional calculus $u(T) = 0$ (cf. [1]). The class C_0, introduced by Sz.-Nagy and Foiaş (cf. [6]), is a familiar class of nonnormal operators on a Hilbert space. In fact, there are numerous theorems concerning the class C_0 in [1] and [6].

The notation and terminology employed herein agree with those in [1], [2], and [6]. For a Hilbert space $\mathcal{H}$ and any operators $T_i \in \mathcal{L}(\mathcal{H})$ $(i = 1, 2)$, we write $T_1 \cong T_2$ if T_1 is unitarily equivalent to T_2.

Note that even if the shift operators are described as various forms, those of the same multiplicity are unitarily equivalent to each other (cf. [3, p. 29] and [4, p. 98]). The main result of this paper is contained in

THEOREM 1. *Let $S^{(n)}$ be a unilateral shift operator of multiplicity n for a positive integer n. Suppose that*

$$S^{(n)} \cong \begin{pmatrix} S^{(n)} & * \\ 0 & E \end{pmatrix} \tag{3}$$

relative to a decomposition $\mathcal{M} \oplus \mathcal{N}$. Then $E \in C_0$.

We expect to demonstrate the utility of Theorem 1 in the theory of dual operator algebras in our future papers stemming from [5].

Let us consider a function $\Theta(\lambda) \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ $(\lambda \in \mathbf{U})$ defined by

$$\Theta(\lambda) = \sum_{k=0}^{\infty} \lambda^k \Theta_k , \tag{4}$$

where $\Theta_k \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ and the series is convergent in the strong (or, equivalently, weak (cf. [6, p. 186])) operator topology. A function $\{\mathcal{K}, \mathcal{H}, \Theta(\lambda)\}$ is called a *bounded analytic function* if there exists $M > 0$ such that $\|\Theta(\lambda)\| \le M$ $(\lambda \in \mathbf{U})$. A contractive analytic function

$$\{\mathcal{K}, \mathcal{H}, \Theta(\lambda)\} \qquad (i.e., \|\Theta(\lambda)\| \le 1, \lambda \in \mathbf{U}) \tag{5}$$

is called *purely contractive* if $\|\Theta(0)a\| < \|a\|$ for all $a \in \mathcal{K}$, $a \neq 0$. We define the *adjoint* $\{\mathcal{H}, \mathcal{K}, \widetilde{\Theta}(\lambda)\}$, by $\widetilde{\Theta}(\lambda) = \Theta(\bar{\lambda})^*$ $(\lambda \in \mathbf{U})$.

Recall that $L^2(\mathcal{H})$ denotes the class of functions $v(t)$ $(0 \le t \le 2\pi)$ with values in $\mathcal{H}$, strongly (or, equivalently, weakly (cf. [6, p. 182])) measurable and such that

$$\int_0^{2\pi} \|v(t)\|^2 \, dt < \infty. \tag{6}$$

Then for any $v \in L^2(\mathscr{H})$, there exists a sequence $\{a_k\}_{-\infty}^{\infty}$ in $\mathscr{H}$ with $\sum_{-\infty}^{\infty} \|a_k\|^2 < \infty$ such that $v(t) = \sum_{-\infty}^{\infty} e^{ikt} a_k$. This means that

$$(7) \qquad \int_0^{2\pi} \|v(t) - \sum_{-m}^{n} e^{ikt} a_k\|^2 \, dt \to 0 \qquad (m, n \to \infty).$$

Let us denote by $H^2(\mathscr{H})$ the class of functions $u(t)$ in $L^2(\mathscr{H})$ such that $u(t) = \sum_{k=0}^{\infty} e^{ikt} a_k$. For any contractive analytic function $\{\mathscr{K}, \mathscr{H}, \mathbf{\Theta}(\lambda)\}$, we define the operator

$$(8a) \qquad \mathbf{\Theta}: L^2(\mathscr{K}) \to L^2(\mathscr{H})$$

by

$$(8b) \qquad (\mathbf{\Theta} v)(t) = \mathbf{\Theta}(e^{it}) v(t) \quad \text{for } v \in L^2(\mathscr{K}),$$

where $\mathbf{\Theta}(e^{it}) = \lim \mathbf{\Theta}(\lambda)$ $(\lambda \to e^{it}$ non-tangentially a.e.)(strongly), and define the operator

$$(9a) \qquad \mathbf{\Theta}_+: H^2(\mathscr{K}) \to H^2(\mathscr{H})$$

by

$$(9b) \qquad (\mathbf{\Theta}_+ u)(\lambda) = \mathbf{\Theta}(\lambda) u(\lambda) \quad \text{for } u \in H^2(\mathscr{K}).$$

The contractive analytic function $\{\mathscr{K}, \mathscr{H}, \mathbf{\Theta}(\lambda)\}$ is called *inner* if $\mathbf{\Theta}(e^{it})$ is an isometry from $\mathscr{K}$ into $\mathscr{H}$ for almost every t or, equivalently, if $\mathbf{\Theta}_+$ is an isometry from $H^2(\mathscr{K})$ into $H^2(\mathscr{H})$; and *-*inner* if the function $\{\mathscr{H}, \mathscr{K}, \widetilde{\mathbf{\Theta}}(\lambda)\}$ is inner.

Let T be a contraction operator on a Hilbert space $\mathscr{H}$. Recall (cf. [6, p. 238]) that the analytic function $\mathbf{\Theta}_T$ defined on $\mathbf{U}$ by

$$(10) \qquad \mathbf{\Theta}_T(\lambda) = \{-T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T\} | \mathscr{D}_T, \qquad \lambda \in \mathbf{U},$$

satisfies

$$(11) \qquad \|\mathbf{\Theta}_T\|_\infty = \operatorname*{ess\,sup}_{\mathbf{T}} \|\mathbf{\Theta}_T(e^{it})\| \le 1,$$

and $\|\mathbf{\Theta}_T(0)x\| < \|x\|$ for all $x \in \mathscr{D}_T$, where

$$(12) \qquad D_T = (I - T^*T)^{1/2} \quad \text{and} \quad \mathscr{D}_T = \overline{(I - T^*T)^{1/2}\mathscr{H}}.$$

The purely contractive analytic function $\{\mathscr{D}_T, \mathscr{D}_{T^*}, \mathbf{\Theta}_T(\lambda)\}$ on $\mathbf{U}$ is called the *characteristic function* of T.

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The invariant subspaces of a unilateral shift $S^{(n)}$ of multiplicity $n < \infty$ are described as follows:

THEOREM 2. *Let $S^{(n)}: H^2(\mathscr{H}) \to H^2(\mathscr{H})$ be a unilateral shift of multiplicity $n < \infty$, where $\dim \mathscr{H} = n$, and let $\mathscr{N}$ be an invariant subspace for $S^{(n)}$. Then there exist a subspace $\mathscr{K}$ of $\mathscr{H}$ and an inner function $\{\mathscr{K}, \mathscr{H}, \Theta(\lambda)\}$ such that $\mathscr{N} = \Theta_+ H^2(\mathscr{K})$. In particular, the space $\mathscr{K}$ can be identified with the space $\mathscr{N} \ominus ((S^{(n)}|\mathscr{N})\mathscr{N})$.*

PROOF. The first part of Theorem 2 is a known result [6, Theorem V.3.3]. Moreover, the second part is implied in the proof of the same result [6, Theorem V.3.3].

If $T \in \mathscr{L}(\mathscr{H})$ and $\mathscr{K}$ is a semi-invariant subspace for T (that is, there exist invariant subspaces $\mathscr{M}$ and $\mathscr{N}$ for T with $\mathscr{N} \subset \mathscr{M}$ such that $\mathscr{K} = \mathscr{M} \ominus \mathscr{N}$), we shall write $T_{\mathscr{K}} = P_{\mathscr{K}} T|\mathscr{K}$ for the compression of T to $\mathscr{K}$, where $P_{\mathscr{K}}$ is the orthogonal projection whose range is $\mathscr{K}$.

Now the proof of Theorem 1 is completed by applying Theorem 3 below.

THEOREM 3. *Under the hypotheses of Theorem 2, let us assume that*

$$\dim(\mathscr{N} \ominus ((S^{(n)}|\mathscr{N})\mathscr{N})) = n. \tag{13}$$

Then the compression $S^{(n)}_{H^2(\mathscr{H})\ominus\mathscr{N}}$ of $S^{(n)}$ to $H^2(\mathscr{H}) \ominus \mathscr{N}$ belongs to the class C_0.

PROOF. The idea of this proof comes from Professor Carl Pearcy. Let us put $\mathscr{M} = H^2(\mathscr{H}) \ominus \mathscr{N}$ and $E \doteq S^{(n)}_{\mathscr{M}}$. Then we can write

$$S^{(n)} = \begin{pmatrix} A & B \\ 0 & E \end{pmatrix} \tag{14}$$

relative to a decomposition $\mathscr{N} \oplus \mathscr{M}$. Now we shall show that $E \in C_0$. It is well known that A is unitarily equivalent to 0 or $S^{(k)}$, for some k with $1 \le k \le n$. Let $\mathscr{K} = \mathscr{N} \ominus A\mathscr{N}$ be the subspace found by Theorem 2 and let $\{\mathscr{K}, \mathscr{H}, \Theta(\lambda)\}$ be the corresponding inner function. If we suppose that $A \cong 0$, then $\mathscr{N} = (0)$ (otherwise, the kernel of $S^{(n)}$ is nontrivial) and $\mathscr{K} = (0)$. So this contradicts the hypothesis that $\dim \mathscr{K} = n$.

Next suppose that $A \cong S^{(k)}$, $1 \le k \le n-1$. Then $\dim \mathscr{K} = k \le n-1$, and this also yields a contradiction. Hence we can assume that $A \cong S^{(n)}$. Since the operator-valued analytic function $\{\mathscr{K}, \mathscr{H}, \Theta(\lambda)\}$ is inner, $\Theta(e^{it})$ is an isometry a.e.. Moreover, since $\dim \mathscr{K} = \dim \mathscr{H} = n < \infty$, $\Theta(e^{it})$ is a unitary operator on $\mathscr{H}$ for almost all t. It follows from the Decomposition Theorem (cf. [6, p. 188]) that there exists a uniquely determined decomposition $\mathscr{K} = \mathscr{K}^\circ \oplus \mathscr{K}'$ and $\mathscr{H} = \mathscr{H}^\circ \oplus \mathscr{H}'$ such that for every fixed λ, $\Theta^\circ(\lambda) = \Theta(\lambda)|\mathscr{X}^\circ$ has its range in $\mathscr{K}^\circ$, that $\{\mathscr{X}^\circ, \mathscr{K}^\circ, \Theta^\circ(\lambda)\}$ is purely contractive analytic function, and that $\{\mathscr{X}', \mathscr{K}', \Theta'(\lambda)\}$ is a unitary constant. Thus, without loss of generality, we can assume

$$(15) \qquad \mathscr{M} = H^2(\mathscr{K}) \ominus \Theta H^2(\mathscr{X}) = H^2(\mathscr{K}) \ominus \Theta_+ H^2(\mathscr{X}) \neq (0).$$

Therefore, according to [6, Proposition 3.2, p. 255], $\Theta(\lambda)$ is not a unitary constant; equivalently, $\Theta(\lambda)$ has the purely contractive part $\Theta^\circ(\lambda)$. Since

$$(16) \qquad \Theta(e^{it}) = \Theta^\circ(e^{it}) \oplus \Theta'(e^{it}) \quad \text{a.e.}$$

and since $\Theta(e^{it})$ is unitary a.e., $\Theta^\circ(e^{it})$ is unitary a.e.. Therefore $\{\mathscr{X}^\circ, \mathscr{K}^\circ, \Theta^\circ(\lambda)\}$ is inner and *-inner. On the other hand, since E is the compression to $H^2(\mathscr{K})\ominus\Theta_+ H^2(\mathscr{X})$ of multiplication by e^{it}, according to [6, Proposition 3.2, p. 255], the characteristic function $\Theta_E(\lambda)$ of the completely nonunitary contraction E coincides with $\{\mathscr{X}^\circ, \mathscr{K}^\circ, \Theta^\circ(\lambda)\}$. According to [6, Proposition 3.5, p. 257], we have $E \in C_{00}$ (that is, $\|E^n x\| \to 0$ and $\|E^{*n} y\| \to 0$ for all $x, y \in \mathscr{M}$) if and only if $\Theta_E(\lambda)$ is inner and *-inner. Hence $E \in C_{00}$. As was noted above, $\Theta(e^{it})$ is unitary a.e.. For such a t, $\Theta(\lambda)$ is invertible for λ sufficiently close to e^{it}, since $\Theta(e^{it}) = \lim \Theta(\lambda)$ as $\lambda \to e^{it}$ non-tangentially a.e. Finally, according to [6, Proposition 6.1, p. 216], $\Theta(\lambda)$ has a scalar multiple. Thus, by [6, Theorem 5.1, p. 265], we have $E \in C_0$. Hence the proof is complete.

For an invariant subspace $\mathscr{N}$ for $S^{(n)}$, $S^{(n)}|\mathscr{N}$ is a unilaterial shift of some multiplicity (cf. [3, Proposition 7.13]). Hence the hypothesis that $\dim(\mathscr{N} \ominus (S^{(n)}|\mathscr{N})\mathscr{N})) = n$, appearing in Theorem 3, means that the multiplicity of $S^{(n)}|\mathscr{N}$ is n.

For $T \in \mathscr{L}(\mathscr{K})$, we write d_T for the defect index of T, that is, $d_T = \dim \mathscr{D}_T$. Recall that $H^\infty(\mathbf{U})$, the class of all bounded analytic functions on U, is identified with H^∞ (cf. [6, p. 101]). The following is an immediate consequence of Theorem 3.

Corollary. *Under the hypotheses of Theorem 3, let $d \in H^\infty$ be defined by setting $d(\lambda)$ equal to the determinant of $\Theta_E(\lambda)$ corresponding to some fixed orthonormal bases of $\mathscr{D}_E$ and $\mathscr{D}_{E^*}$. Then $d(E) = 0$.*

Proof. Without loss of generality, we assume that E is nontrivial. Since $E \in C_{00}$, it follows from [6, Theorem 1.2, p. 59] that $1 \leq d_E = d_{E^*}$. Moreover, since $d_{E^*} \leq d_{S^{(n)*}} = n$, using [6, Theorem 5.2, p. 266], we have $d(E) = 0$. Hence the proof is complete.

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References

[1] H. Bercovici, *Operator theory and arithmetic in* H^∞ Math. Surveys and Monographs, No. 26, (Amer. Math. Soc., Providence, R.I., 1988).

[2] A. Brown and C. Pearcy, *Introduction to operator theory.* I, *Elements of functional analysis* (Springer-Verlag, New York, 1977).

[3] J. Conway, *Subnormal operators* (Pitman, Boston, 1981).

[4] K. Hoffman, *Banach spaces of analytic functions* (Prentice-Hall, Englewood Cliffs, N.J., 1962).

[5] I. Jung, *Dual operator algebras and the classes* $A_{m,n}$ (Ph.D. Thesis, Univ. of Michigan, 1989).

[6] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on the Hilbert space* (North Holland Akademiai Kiado, Amsterdam/Budapest, 1970).

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